Reduced Modeling of Impedance Networks. Application to Supervision/diagnosis

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Abstract—Having especially in view application to fuel cell systems, we study in this paper the reduced modeling of impedance networks. The networks under study are constituted by a set of approximately identical (sub-)cells, coupled by series-parallel electrical links. In order to detect and diagnose the appearance of disparities in the behavior of the latter (coming, say, from aging or degradations), the description of the electrical behavior of the whole system by a "mean cell" is not sufficient. The main contribution of the paper is to provide a more involved approximation of the global network impedance function, including corrective terms characterizing the dispersion with respect to the average behavior. It is then natural an attempt to identify quantities able to describe the average behavior and the dispersion, in order to use them as alert data for supervision and diagnosis. As an illustrative example, the corresponding identification problem for the reduced model is studied in more details in the case where the individual impedance functions are first-order transfers.

Keywords: Model/Controller reduction; Identification; Energy systems

I. INTRODUCTION

Alternative solutions to produce energy in a clean and efficient way are widely explored nowadays. Among them, fuel cell systems and batteries are intensively developed, both for stationary and mobile applications, as they allow for highpower density energy storage. Their implementation requires special care, and ensuring viability, energetic efficiency and robustness under a variety of environmental conditions and a wide operating range requires precise monitoring.

Impedance spectroscopy is a powerful method of characterizing many of the electrical properties of materials and their interfaces, and thus to study (natural or engineered) systems whose operation relies on these properties. Basically, it consists in applying electrical stimuli and recording the induced current/tension relation. Impedance spectroscopy has proved particularly useful for characterizing the behavior of electrochemical power sources, which are the place of strongly coupled chemical and electrical (and to a lesser extent mechanical and thermal) phenomena. It also yields other areas of interest to the engineer, such as characterization of materials, study of the corrosion and others [1], but also to the clinician [3] or the microbiologist [8] with the study of living cells.

Extensive work has been made to understand and to provide models of energy-conversion systems (see e.g. [4], [5] for fuel cell systems) as differential systems. The complexity of such models range from 3D partial differential equations involving multi-phases phenomena, interaction liquid/solid and thermal effects, to simple models based on ordinary differential equations (ODE). The latter are especially interesting for the control scientist purposes, including the tasks of supervision and diagnosis in which we are interested here. Notice that these models, although usually nonlinear, may be used to provide analytical expressions for the impedance function (see below in Section II).

Usually, the ODE models end up with a vision "at the cell level": the description of the chemical reactions and electrical phenomena involved is done under the assumption that quantities such as concentrations of species, current densities, temperatures or physical characteristics are *homogeneous* in the whole device. However, models resulting from this point of view may be insufficient for supervision and diagnosis purposes. Here, the underlying hypothesis is that the failures we want to supervise appear locally in the fuel cell system, due to modifications e.g. of the membrane electrical, chemical or mechanical properties due to drying, flooding, membrane poisoning, aging... So the degraded modes are typically accompanied by a loss of homogeneity, which can be reversible or not.

Having especially in mind the application to fuel cell systems, our goal in this paper is to explore the issue of *reduced modeling* of the impedance function of a *series-parallel network of cells*. This structure is considered in order to account for the disparities between different cells — the latter being put electrically in series —, as well as for the possible heterogeneities within the cells — giving rise to the parallel aspect. The underlying hypothesis here is that loss of synchronicity induces efficiency drop: this is of course this last aspect which is ultimately of interest here.

Our contribution here is an attempt to describe the impedance of a network of approximately identical cells in a more precise way than the usual "mean cell" models, without however changing drastically the complexity of the model. Moreover, we exhibit new quantities to be used for monitoring and diagnosis as a clue of efficiency loss. While aimed preferentially at fuel cell systems, our study here stays at an abstract level, as we're only interested in the reduced modeling aspects of an impedance network.

The paper is organized as follows. It is first shown in Section II how a differential model yields, via harmonic balance, to compute impedance function. The series-parallel model of network we aim at studying is presented in Section III. Based on an expansion procedure, the reduced impedance

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model is introduced in Section IV. Last, it is shown in Section V how the previous computations lead to identification of mean parameter values and to quantities characterizing the dispersion of the latter, just like "variances" would. For simplicity, the computations are conducted for an illustrative example, while for real applications physical model should be used. In agreement with the simplest impedance functions encountered in practice [1], the class of first-order transfers is considered here.

Notations. In the sequel, derivatives with respect to the *k*-th component θ_k of the parameter vector θ is denoted ∂_k , and second derivatives with respect to θ_k, θ_l : ∂_{kl} . Thus, the gradient (with respect to θ) of $Z(\theta; s)$ is the vector $\nabla Z(\theta; s) = \nabla_{\theta} Z(\theta; s) = (\partial_k Z(\theta; s))_k$; and H(Z), the Hessian of Z, is the matrix $(\partial_{kl} Z(\theta; s))_{kl}, \partial_{kl} Z \doteq \frac{\partial^2 Z(\theta; s)}{\partial \theta_k \partial \theta_l}$.

II. IMPEDANCE OF ELEMENTARY CELL

The, possibly nonlinear, input/output behavior of any elementary cell of the network is given by

$$\dot{x} = f(\theta; x, u), \quad y = g(\theta; x, u)$$
 (1)

Here as usual, u, x, y are respectively the input, state and output variables. All are supposed finite-dimensional, and uand y are moreover assumed *scalar*. As a matter of fact, uwill have in the sequel the meaning of a *current*, while y is a measured *potential*. Last, θ is a *vector of parameters*, of finite-dimension. Typically based on balance equations and behavioral laws, f and g are considered here as known.

The most common and standard method in impedancemetry [1] consists in feeding (1) with sinusoidal monochromatic inputs $u = u_0 + u_1 \cos \omega t$ for various values of the frequency ω . It is assumed that, after possible transients, the state and output of system (1) oscillates periodically, at the same frequency ω . This is indeed quite usually what happens to stable systems under forced oscillations. For small u_1 , or if the system has low-pass filter properties, the state xand the output y have essentially the same form: $x(t) \sim$ $x_0 + x_1 \cos(\omega t + \varphi_x)$ and $y(t) \sim y_0 + y_1 \cos(\omega t + \varphi_y)$.

The impedance is then the complex I/O gain at the fundamental frequency ω . This is reminiscent of course of *harmonic balance method* [2]. More precisely, according to this method the coefficients are computed from:

$$x_0 \doteq \frac{\omega}{2\pi} \int_0^{2\pi/\omega} x(t) dt, \quad x_1 e^{j\varphi_x} \doteq \frac{\omega}{\pi} \int_0^{2\pi/\omega} x(t) e^{i\omega t} dt$$

and similarly for y_0, y_1 (as can be seen, second and higher terms have been ignored). When $|x_1| \ll |x_0|$, which is usually the case in impedancemetry, first-order expansion shows that (3) holds. The first formula in (3) indicates how to deduce the steady-state values of x_0 , and then of y_0 , from u_0 ; while the second one (in which the gradient with respect to u and x are denoted by ∇_u, ∇_x) provides analytically the value of the impedance.

Thus, one may consider at least formally, that from the nonlinear model (1) can be deduced the impedance function $Z(\theta; s)$ in (3) for an elementary cell.

III. MODELING A SERIES-PARALLEL NETWORK OF CELLS

We now present the complex network under study, which is supposed to represent a whole fuel cell stack. The model of the global impedance of the network is written $Z_{\rm mod}(s)$ and given by

$$Z_{\text{mod}}(s) = \frac{1}{m} \sum_{j=1}^{m} \left(\frac{1}{n} \sum_{i=1}^{n} Z(\theta_{ij}; s)^{-1} \right)^{-1}$$
(2)

see Figure 1. In this formula, the variable j allows to index the different cells located in series electrically. We would like to have the ability to describe also internal disparities in the parameter distribution inside each cell of the stack. This is why is introduced the variable i. Thus, each cell is seen as a population of *sub-cells* put in parallel electrically.

Notice that in (2), θ_{ij} is the *vector* of parameters corresponding to the sub-celle *i* in the cell *j*. When needed, the *k*-th component of this vector will be denoted $\theta_{k,ij}$.

IV. AN EXPANSION-BASED FREQUENCY DOMAIN REDUCED MODEL

A. The reduced model formula

We begin by presenting the formula of the reduced model, in formula (4). The latter, corresponding to the complete model of Figure 1, is shown on Figure 2.

Recall that $H(Z)(\theta)$ (resp. $H(Z^{-1})(\theta)$) represents Hessian matrix with respect to the parameter vector θ of $Z(\theta) = Z(\theta; s)$ (resp. of its inverse $Z(\theta)^{-1}$). By definition, in (4) $\bar{\theta}$ is defined as the mean-value of θ on the whole network, and $\bar{\theta}_{\eta}$ is the mean-value of θ on the cell η . A crucial point is that $Z_{\rm red}(s)$ depends only upon some mean-values and variances related to the distribution of the parameters θ , while $Z_{\rm mod}(s)$ was a function of the value of the parameters at every point (in each sub-cell) of the network. In this sense, $Z_{\rm red}(s)$ is really a reduced model of $Z_{\rm mod}(s)$.

It is evident from these formulas that, when the corrective terms (bearing Hessians H(Z) and $H(Z^{-1})$) are null, then the reduced impedance is just the impendance $Z(\bar{\theta}; s)$ of the mean-cell, corresponding to the average parameter values.

B. Relation with the exact model

The choice of the reduced model above comes from expanding $Z(\theta; s)$ around the value of $\overline{\theta}$ in the expression of the exact impedance function. The formal relation between the global impedance and its approximation is provided now.

Proposition 1. Assume the function Z is analytic with respect to θ . The functions $Z_{mod}(s)$ and $Z_{red}(s)$ are equal up to moments of the form

$$\sum_{j} \left(\sum_{i} (\theta_{ij} - \bar{\theta})^p \right)^q (\bar{\theta}_j - \bar{\theta})^r, \quad p, q, r \ge 0, \ pq + r \ge 3 \quad \Box$$

We now illustrate the reduction procedure in a simple illustrative example, that will be considered again latter.



Fig. 1. The series-parallel network corresponding to $Z_{
m mod}(s)$ in (2).



Fig. 2. Principle of the reduced model (4) of (2). The approximate impedance is equal to the impedance of a sub-cell corresponding to the mean-value of the parameters, in parallel and in series with two corrective terms. The latter depend upon momentums of second order characterizing the heterogeneities in the parameter distribution, see formula (4).

$$f(\theta; x_0, u_0) = 0, \quad y_0 = g(\theta; x_0, u_0)$$
 (3a)

$$Z(\theta;s) \doteq \frac{y_1 e^{j\varphi_y}}{u_1} = \nabla_x g(\theta;x_0,u_0) \left(sI - \nabla_x f(\theta;x_0,u_0)\right)^{-1} \nabla_u f(\theta;x_0,u_0) + \nabla_u g(\theta;x_0,u_0) \ . \tag{3b}$$

$$Z_{\rm red}(s) = \frac{1}{2m} \sum_{j=1}^{m} (\bar{\theta} - \bar{\theta}_j)^{\mathsf{T}} H(Z)(\bar{\theta})(\bar{\theta} - \bar{\theta}_j) + Z(\bar{\theta}; s) \left(1 + \frac{Z(\bar{\theta}; s)}{2mn} \sum_{j=1}^{m} \sum_{i=1}^{n} (\theta_{ij} - \bar{\theta}_j)^{\mathsf{T}} H(Z^{-1})(\bar{\theta})(\theta_{ij} - \bar{\theta}_j) \right)^{-1}$$
(4a)

$$\bar{\theta}_j \doteq \frac{1}{n} \sum_{i=1}^n \theta_{ij}, \qquad \bar{\theta} \doteq \frac{1}{m} \sum_{j=1}^m \bar{\theta}_j . \tag{4b}$$

$$\alpha_{kl} \doteq \frac{1}{m} \sum_{j=1}^{m} (\bar{\theta}_k - \bar{\theta}_{k,j}) (\bar{\theta}_l - \bar{\theta}_{l,j}), \qquad \beta_{kl} \doteq \frac{1}{mn} \sum_{j=1}^{m} \sum_{i=1}^{n} (\theta_{k,ij} - \bar{\theta}_{k,j}) (\theta_{l,ij} - \bar{\theta}_{l,j})$$
(5a)

$$\bar{\theta}_{k,j} \doteq \frac{1}{n} \sum_{i=1}^{n} \theta_{k,ij}, \qquad \bar{\theta}_k \doteq \frac{1}{m} \sum_{j=1}^{m} \bar{\theta}_{k,j}, \qquad k, l = 1, 2, 3$$
 (5b)

$$Z_{\rm red}(s) = \frac{\bar{\theta}_2 \alpha_{33}}{\tilde{s}^3} - \frac{\alpha_{23}}{\tilde{s}^2} + \frac{(\bar{\theta}_1^2 + \beta_{11})\tilde{s}^3 + 2(\bar{\theta}_1\bar{\theta}_2 + \beta_{12})\tilde{s}^2 + (\bar{\theta}_2^2 + \beta_{22} - 2\bar{\theta}_2\beta_{13} + \bar{\theta}_1\beta_{23})\tilde{s} - (\bar{\theta}_1\bar{\theta}_2\beta_{33} + \bar{\theta}_2\beta_{23})}{(\bar{\theta}_1\bar{\theta}_2\beta_{33} + \bar{\theta}_2\beta_{23})}$$
(6)



Fig. 3. Comparison between the exact impedance curves (in blue) and the approximations obtained with the single (mean) cell model $Z(\bar{\theta}; s)$ (in green) and the complete reduced model (in red). For details, refer to Section IV-B.

Example 1 (First-order impedance.). Consider the impedance class corresponding to simple RC circuits, given by the first-order transfers:

$$Z(\theta, s) \doteq \theta_1 + \frac{\theta_2}{s + \theta_3} \tag{7}$$

where $\theta_3 > 0$ (for stability reasons) and $\theta_1, \theta_2 \ge 0$ (passivity). Here, the parameters θ are thus a priori chosen as a set of variables allowing to represent first order impedance functions. Generally speaking, the parameters can also be chosen directly as physical quantities entering in an intricate way in some of the coefficients of the impedance function.

For k, l = 1, 2, 3, consider the quantities defined in (5). They are first and second order moments, depicting preferability series or parallel heterogeneities. Computations show that the impedance model given in (7) writes as in (6), where $\tilde{s} \doteq s + \theta_3$ for brevity.

Figure 3 is computed for a "10×10 network" (that is: m = n = 10), with $\theta_{1,ij} = 9 \pm 30\%$, $\theta_{2,ij} = 6 \pm 30\%$, $\theta_{3,ij} = 1 \pm 30\%$ and randomly generated variations. Figure 3 shows in the Nyquist plane the exact impedance curve (2) (blue); the impedance curve obtained by use of the meanvalue term $Z(\bar{\theta}; s)$ only (with the exact values $\bar{\theta}$; green); and the complete reduced model impedance (4), which for 1storder sub-cell impedance reads as (6) (with the exact values of $\bar{\theta}$, α , β ; red). The approximation of $Z_{mod}(s)$ by $Z_{red}(s)$ is rather sharp (the maximal error in real or imaginary part is less than 10^{-2} in the whole frequency range), and always much better than with the mean-value term only.

V. IDENTIFYING THE REDUCED MODEL

A. Methodology

One assumes that impendance spectroscopy measurements are available, usually a finite number of points $Z_{\text{mes}}(j\omega), \omega \in \Omega$ of the Nyquist plot. Our purpose is to face the issue of identifying the mean parameter values $\bar{\theta}$, together with the parameter dispersions given by α and β , see (5). In view of (6), there are indeed only *eleven* scalar parameters: $\bar{\theta}_1, \bar{\theta}_2, \bar{\theta}_3, \alpha_{23}, \alpha_{33}, \beta_{11}, \beta_{12}, \beta_{13}, \beta_{22}, \beta_{23}, \beta_{33}$.

We propose an identification procedure in two steps.

• First, the measurements are used to fit optimally (in a sense to be made precise) the parameters of an impedance function chosen in an appropriate class. This issue — fitting optimally some parametrized curve to measurements — appeals to classical questions in identification and is not specifically explored further here.

• The second step consists in considering the identification of $\bar{\theta}, \alpha$ and β itself, that is the issue of existence and uniqueness of values of these variables for which the corresponding function $Z_{\rm red}(s)$ equals the fitted curve. Of course, the choice of the class of functions is made on the basis of the structure of $Z_{\rm red}(s)$, and thus ultimately upon the impedance model at the (sub-)cell scale.

From now on, we consider *rational* impedance function, by assuming that the (scalar) function Z writes as:

$$Z(\theta; s) \doteq \frac{n(\theta; s)}{d(\theta; s)} \tag{8}$$

where $n(\theta; s)$ and $d(\theta; s)$ are polynomial with respect to the Laplace variable s.

B. Decoupling between α and β

Let us first state a technical result, whose proof is omitted.

Lemma 2. The Hessian $H\left(\frac{n}{d}\right)$ of a ratio of functions $\frac{n}{d}$, is equal to $\frac{1}{d}H(n) - \frac{n}{d^2}H(d) - \frac{1}{d^2}\left(\nabla n \cdot \nabla d^{\mathsf{T}} + \nabla d \cdot \nabla n^{\mathsf{T}}\right) + 2\frac{n}{d^3}\nabla d \cdot \nabla d^{\mathsf{T}}$.

We can then deduce the following result.

Proposition 3. For Z as in (8), $Z_{red}(s)$ defined by (4) writes as in (9).

Keeping in mind that the unknowns of the identification problem are the vector of mean value parameters $\bar{\theta}$ and the coefficients of α , β , one sees that the previous form is important: for rational functions as in (8), the *numerator* depends only upon $\bar{\theta}$ and α , while the *denominator* depends only upon $\bar{\theta}$ and β . This property is exemplified and explained now, based on Example 1.

Example 1 (continued). Reducing to common denominator, (6) leads to: $Z_{red}(s) = \frac{n(s)}{d(s)}$, with $d(s) \doteq (\bar{\theta}_1^2 + \beta_{11})\tilde{s}^3 + 2(\bar{\theta}_1\bar{\theta}_2 + \beta_{12})\tilde{s}^2 + (\bar{\theta}_2^2 + \beta_{22} - 2\bar{\theta}_2\beta_{13} + \bar{\theta}_1\beta_{23})\tilde{s} - (\bar{\theta}_1\bar{\theta}_2\beta_{33} + \bar{\theta}_2\beta_{23})$ and $n(s) \doteq (\bar{\theta}_2\alpha_{33} - \alpha_{23}\tilde{s}) d(s) + \tilde{s}^3(\bar{\theta}_1\tilde{s} + \bar{\theta}_2)^3$. The denominator of $Z_{red}(s)$ is a polynomial of degree deg d = 3 in $\tilde{s} = s + \bar{\theta}_3$ (and thus in s), and the numerator a polynomial of degree deg n = 6. It thus seems reasonable to attempt to identify its coefficients to a given expression of the form:

$$\frac{\tilde{s}^6 + y_1\tilde{s}^5 + y_2\tilde{s}^4 + y_3\tilde{s}^3 + y_4\tilde{s}^2 + y_5\tilde{s} + y_6}{\tilde{s}^3(x_0\tilde{s}^3 + x_1\tilde{s}^2 + x_2\tilde{s} + x_3)} .$$
(11)

Here the constants x_0, x_1, x_2, x_3 and $y_1, y_2, y_3, y_4, y_5, y_6$ are supposed to have been deduced from the impedance spectroscopy measurements, as well as $\bar{\theta}_3$, the triple root of the denominator (recall that for simplicity the variable $\tilde{s} = s + \bar{\theta}_3$ has been introduced in place of s). This stage is

$$Z_{\rm red}(s) = n^3 \left(n^2 + \frac{1}{2} \sum_{k,l} \beta_{kl} \left\{ n^2 \partial_{kl} d - nd \ \partial_{kl} n - n \left(\partial_k d \ \partial_l n + \partial_k n \ \partial_l d \right) + 2d \ \partial_k n \ \partial_l n \right\} \right)^{-1} + \frac{1}{2} \sum_{k,l} \alpha_{kl} \left\{ \frac{1}{d} \partial_{kl} d - \frac{n}{d^2} \partial_{kl} d - \frac{1}{d^2} \left(\partial_k d \ \partial_l n + \partial_k n \ \partial_l d \right) + 2 \frac{n}{d^3} \partial_k d \ \partial_l d \right\} .$$
(9)
$$\left(n^2 + \frac{1}{2} \sum_{k,l} \beta_{kl} \left\{ n^2 \partial_{kl} d - nd \ \partial_{kl} n - n \left(\partial_k d \ \partial_l n + \partial_k n \ \partial_l d \right) + 2d \ \partial_k n \ \partial_l n \right\} \right) d(s)^3$$
(10a)
$$n^3 + \frac{1}{2} \left(n^2 + \frac{1}{2} \sum_{k,l} \beta_{kl} \left\{ n^2 \partial_{kl} d - nd \ \partial_{kl} n - n \left(\partial_k d \ \partial_l n + \partial_k n \ \partial_l d \right) + 2d \ \partial_k n \ \partial_l n \right\} \right) \times \sum_{k,l} \alpha_{kl} \left\{ d^2 \ \partial_{kl} n - nd \ \partial_{kl} d - d \ \left(\partial_k d \ \partial_l n + \partial_k n \ \partial_l d \right) + 2n \ \partial_k d \ \partial_l d \right\} .$$
(10b)

supposed to be the first step of the identification procedure (see Section V-A).

Corresponding to the ten coefficients in (11), ten identities may thus be deduced, rendering obvious the decoupling property mentioned above. For the denominator:

$$\frac{1}{\overline{\theta}_1} + \frac{\beta_{11}}{\overline{\theta}_1^3} = x_0 \tag{12a}$$

$$2\left(\frac{1}{\bar{\theta}_1}\frac{\bar{\theta}_2}{\bar{\theta}_1} + \frac{\beta_{12}}{\bar{\theta}_1^3}\right) = x_1 \tag{12b}$$

$$\frac{1}{\bar{\theta}_1} \left(\frac{\bar{\theta}_2}{\bar{\theta}_1}\right)^2 + \frac{\beta_{22}}{\bar{\theta}_1^3} - 2\frac{\bar{\theta}_2}{\bar{\theta}_1}\frac{\beta_{13}}{\bar{\theta}_1^2} + \frac{\beta_{23}}{\bar{\theta}_1^2} = x_2 \qquad (12c)$$

$$\frac{\theta_2}{\bar{\theta}_1}\frac{\beta_{33}}{\bar{\theta}_1} + \frac{\theta_2}{\bar{\theta}_1}\frac{\beta_{23}}{\bar{\theta}_1^2} + x_3 = 0 , \qquad (12d)$$

and for the numerator:

$$3\frac{\bar{\theta}_2}{\bar{\theta}_1} = y_1 \tag{13a}$$

$$3\left(\frac{\bar{\theta}_2}{\bar{\theta}_1}\right)^2 - \frac{\alpha_{23}}{\bar{\theta}_1^3} x_0 = y_2 \tag{13b}$$

$$\left(\frac{\bar{\theta}_2}{\bar{\theta}_1}\right)^3 + \frac{\bar{\theta}_2}{\bar{\theta}_1}\frac{\alpha_{33}}{\bar{\theta}_1^2}x_0 - \frac{\alpha_{23}}{\bar{\theta}_1^3}x_1 = y_3$$
(13c)

$$\frac{\theta_2}{\overline{\theta_1}} \frac{\alpha_{33}}{\overline{\theta_1}^2} x_1 - \frac{\alpha_{23}}{\overline{\theta_1}^3} x_2 = y_4$$
(13d)

$$\frac{\overline{\theta}_2}{\overline{\theta}_1} \frac{\alpha_{33}}{\overline{\theta}_1^2} x_2 - \frac{\alpha_{23}}{\overline{\theta}_1^3} x_3 = y_5$$
(13e)

$$\frac{\theta_2}{\bar{\theta}_1} \frac{\alpha_{33}}{\bar{\theta}_1^2} x_3 = y_6 . \tag{13f}$$

C. Remarks on the structure of the identification problem

Referring to (9), one sees clearly that, for the rational impedance of the form (8), the expression of $Z_{red}(s)$ is highly structured. The (common) denominator and the numerator are given by formulas (10). The degree (in s) of the former is at most equal to $4 \deg d+2 \deg n$, and the degree of the latter to $\deg n+2 \max\{\deg n, \deg d\}$. (Recall that we are

considering degrees with respect to s, while the derivatives written with the symbol ∂ are achieved with respect to the parameters.)

The first factor in the second term of the numerator comes from the denominator: its coefficients can be considered as known. Using this trick, one sees that the obtained equations are indeed *always linear with respect to* α *and* β .

Concerning the issue of their under- or over-determination, it does not seem evident to rule on in general. However, the previous example shows a very particular additional structure.

Example 1 (continued). *Detailed computations show that from the six numerator formulas in* (13) *are deduced the following formulas*

$$\frac{\theta_2}{\overline{\theta_1}} = \frac{y_1}{3} \tag{14a}$$

$$\frac{\alpha_{33}}{\bar{\theta}_1^2} = \frac{3y_6}{x_3y_1} \tag{14b}$$

$$\frac{\alpha_{23}}{\bar{\theta}_1^3} = \frac{x_2}{x_3} \frac{\theta_2}{\bar{\theta}_1} \frac{\alpha_{33}}{\bar{\theta}_1^2} - \frac{y_5}{x_3} = \frac{x_2 y_6}{x_3^2} - \frac{y_5}{x_3} .$$
(14c)

as well as three compatibility relations

$$\frac{1}{3}x_3^2y_1^2 + x_0x_3y_5 = x_3^2y_2 + x_0x_2y_6$$
(15a)

$$\frac{1}{27}x_3^2y_1^3 + x_0x_3y_6 + x_1x_3y_5 = x_3^2y_3 + x_1x_2y_6$$
(15b)

$$x_1 x_3 y_6 + x_2 x_3 y_5 = x_3^2 y_4 + x_2^2 y_6 \tag{15c}$$

The violation of the latter (or the impossibility of finding x_r, y_r in (11) nicely fitting experimental data and fulfilling them) can be the mark that 'something goes wrong'. Here, they are integrated to the first step of the identification procedure: one directly looks for x_r, y_r which better fit the experimental data among those which fulfill the previous algebraic constraints. However, the latter are nonlinear, and generally speaking, this removes any guarantee that the corresponding optimization problem is nicely solvable (in

Quantities	Exact values	Values computed	Relative errors
$ heta_3$	1.0757	1.0485	2.5%
$\frac{\theta_2}{\overline{\theta}_1}$	0.6611	0.6758	2.2%
$\left[\frac{1}{\bar{\theta}_1} + \frac{\beta_{11}}{\bar{\theta}_1^3}\right]$	0.1105	0.1119	1.3%
$\frac{\theta_2}{\bar{\theta}_1^2} + \frac{\beta_{12}}{\bar{\theta}_1^3}$	0.0724	0.0763	5.4%
$\frac{\theta_2^2}{\overline{\theta}_1^3} + \frac{\beta_{22}}{\overline{\theta}_1^3} - 2\frac{\theta_2\beta_{13}}{\overline{\theta}_1^3} + \frac{\beta_{23}}{\overline{\theta}_1^2}$	0.0495	0.0475	4.0%
$\boxed{\frac{\beta_{33}}{\bar{\theta}_1} + \frac{\beta_{23}}{\bar{\theta}_1^2}}$	0.0032	0.0039	22%

Fig. 4. Exact and computed values of six of the eight identifiable quantities.

particular, there is in general no guarantee at all on the absence of local minima).



Fig. 5. Comparison between the exact data and the identified impedance curve.

Based on the data already used at the end of Section IV-B, a comparison is provided in Figure 5 between the Nyquist plot of the exact model (as in Figure 3; red), with the function of type (11) (blue), where the x_r, y_r are chosen to minimize the sum of Euclidian distances $\sum_{\omega \in \Omega} ||Z_{\text{mes}}(j\omega) - Z_{\text{red}}(j\omega)||^2$ while fulfilling the compatibility relations (15). The set Ω is the frequency set $\{10^{\omega} : \omega = -2, -19, -18, -29, 3\}$

the frequency set $\{10^{\omega} : \omega = -2, -1.9, -1.8, \dots, 2.9, 3\}$. Very good accordance is obtained.

The denominator (12) furnishes four identifiable quantities. Overall, eight quantities may be identified from the measurements, namely: $\bar{\theta}_3$, $\frac{\bar{\theta}_2}{\bar{\theta}_1}$, $\frac{\alpha_{23}}{\bar{\theta}_1^3}$, $\frac{\alpha_{23}}{\bar{\theta}_1^3}$, $\frac{1}{\bar{\theta}_1} + \frac{\beta_{11}}{\bar{\theta}_1^3}$, $\frac{\bar{\theta}_2}{\bar{\theta}_1^2} + \frac{\beta_{12}}{\bar{\theta}_1^3}$, $\frac{\bar{\theta}_2}{\bar{\theta}_1^3} + \frac{\beta_{23}}{\bar{\theta}_1^3} - 2\frac{\bar{\theta}_2\beta_{13}}{\bar{\theta}_1^3} + \frac{\beta_{23}}{\bar{\theta}_1^2}$ and $\frac{\beta_{33}}{\bar{\theta}_1} + \frac{\beta_{23}}{\bar{\theta}_1^2}$.

However, the parameters $\overline{\theta}$, α , β themselves cannot be identified, there remains under-determination.

D. Numerical application: identification for Example 1

The identification is conducted as presented above, on the data already generated in Section IV-B, see Table 4. As can be seen, the comparison between identified and exact values is very good for five quantities. These preliminary results show already the interest of the reduced modeling method.

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VI. CONCLUSION

Reduced modeling of impedance network made up of parallel/series coupling of almost identical sub-cells has been studied. A method based on simple expansion formula is proposed, and the reduced model is obtained as a parallel/series assembly of three terms, a term corresponding to a 'mean cell' behavior, and two terms taking account series, respectively parallel, disparities in the sub-cell parameter distribution. Based on identification of the reduced model, the method, while still at an initial stage, may be used for supervision/diagnosis.

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